MATH350 - Exemplo 3.3

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Exemplo 3.3

$$X \sim \Gamma(\alpha, \beta)$$

$$\theta = (\alpha, \beta) \qquad \text{e} \qquad \Omega = (0, \infty) \times (0, \infty)$$

$$f(x, \theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{\beta x}$$

onde

$$\Gamma(\alpha) = \int_0^\infty s^{\alpha-1} e^{-\beta x} ds \qquad \text{e} \qquad E(X) = \frac{\alpha}{\beta}$$

$$
\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} \left\{ \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{\beta x} \right\} \\
&= \left(\frac{\beta^{\alpha}}{\Gamma(\alpha)} \right)^{n} e^{-\beta \sum x} \prod_{i=1}^{n} x^{\alpha-1}
\end{aligned}
$$

$$
l(\theta) = n\alpha log(\beta) - nlog[\Gamma(\alpha)] - \beta \sum_{i=1}^{n} x_i + (\alpha - 1) \sum_{i=1}^{n} log(x_i)
$$

considerando

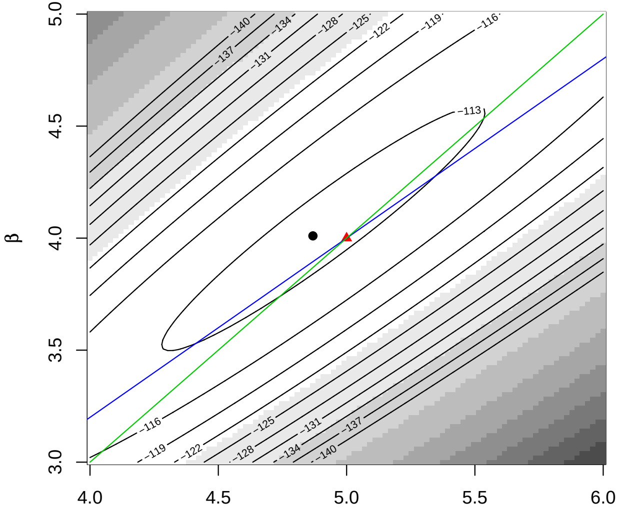
$$n = 25, \ \alpha = 4 \ e \ \beta = 5$$

temos

$$E(X) = \frac{\alpha}{\beta} = \frac{4}{5}$$

Essa esperança pode ser demonstrada pela definição.

$$E(X) \int^{D} x.f(x)dx$$


β
5.0
4.5
4.0
3.5
3.0
4.0
4.5
5.0
5.5
6.0
-140
-137
-134
-131
-128
-125
-122
-119
-116
-113

Exemplo 3.3 (cont) Fazendo a derivada segunda

$$\frac{\delta l(\theta)}{\delta \alpha} = nlog(\beta) + \sum_{i=1}^{n} log(x_i) - n\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$$

$$\frac{\delta l(\theta)}{\delta \beta} = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} x_i$$

$$\frac{n\hat{\alpha}}{\hat{\beta}} - \sum_{i=1}^{n} x_i = 0 \rightarrow \hat{\beta} = \frac{\hat{\alpha}}{\bar{X}}$$

substituindo em $\frac{\delta l(\theta)}{\delta \beta}$

$$nlog(\hat{\alpha}) - nlog(\bar{X}) + \sum_{i=1}^{n} log(x_i) - n\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$$

que requer solução numérica

segunda derivadas

$$
\begin{aligned}
\frac{\delta^2 l(\hat{\theta})}{\delta\alpha^2} &= -n\left[\frac{\Gamma''(al\hat{p}ha)}{\Gamma(al\hat{p}ha)} - \left(\frac{\Gamma'(al\hat{p}ha)}{\Gamma(al\hat{p}ha)}\right)^2\right] \\
\frac{\delta^2 l(\theta)}{\delta\beta} &= -\frac{n\hat{\alpha}}{\beta^2} \\
\frac{\delta^2 l(\hat{\theta})}{\delta\alpha\delta\beta} &= \frac{n}{\hat{\beta}}
\end{aligned}
$$

Exemplo 3.3 (cont)
i.e. $X_i \sim \Gamma(\alpha, \beta)$.

$$U(\theta) = \left\{ n \log \beta + \sum_{i=1}^{n} \log X_i - n\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}, \frac{n\alpha}{\beta} - \sum_{i=1}^{n} x_i \right\}^T$$

e

$$I_O(\theta) = I_E(\theta) = \begin{pmatrix} n[\Gamma''(\alpha)/\Gamma(\alpha) - (\Gamma(\alpha)/\Gamma(\alpha))^2] & -n/\beta \\ -n/\beta & n\alpha/\beta \end{pmatrix}$$

i.e. series expandida multi-variavel de Taylor

$$l(\theta) = l(\hat{\theta}) + (\theta - \hat{\theta})^T U(\hat{\theta}) - \frac{1}{2}(\theta - \hat{\theta})^T I_O(\theta^*)(\theta - \hat{\theta})$$

para

$$\|\theta^* - \theta\| \leq \|\hat{\theta} - \theta\| \qquad (4.1)$$

$$U(\theta) = U(\hat{\theta}) - (\theta - \hat{\theta})^T I_O(\theta^+)$$

para

$$\|\theta^+ - \theta| \leq |\hat{\theta} - \theta\| \qquad (4.2)$$

Para Deviance aplicando equaçao (4.1)

$$\begin{aligned} D(\theta) &= 2\left[l(\hat{\theta}) - \{l(\hat{\theta}) + (\theta - \hat{\theta})^T U(\hat{\theta}) + \frac{1}{2}(\theta - \hat{\theta})^T I_O(\theta^*)(\theta - \hat{\theta})\}\right] \\ &= -(\theta - \hat{\theta})^T I_O(\theta^*)(\theta - \hat{\theta}) \\ &\approx -(\theta - \hat{\theta})^T I_O(\hat{\theta})(\theta - \hat{\theta}) \end{aligned}$$